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A Remark on Papers by Pixton and Oliveira: Genericity of Symplectic Diffeomorphisms of S^2 with Positive Topological Entropy

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We prove the existence of an open and dense subset of maps $f \in Diff^{\infty}_{\omega}(S^2)$ which have positive topological entropy. It follows that these maps have infinitely many hyperbolic periodic points and an exponential growth rate of hyperbolic periodic points. The proof is an application of Pixton's theorem.

KEY WORDS: Genericity; chaos; topological entropy.

Topological entropy characterizes the total exponential orbit complexity of a map with a single number (see ref. 4 for definitions and properties). Topological entropy, especially in low-dimensional cases, provides a wealth of qualitative structural information about the system, including the growth rate of the number of periodic orbits,[(2)] existence of *large* horseshoes,[(3)] and the growth rate of the volume of cells of various dimensions.[(13)] Any map which possesses a horseshoe, i.e., some power of the map is topologically conjugate to a Bernoulli shift, has positive topological entropy. Katok[(2)] has shown that the converse is true for surface diffeomorphisms. Hence surface diffeomorphisms having positive topological entropy exhibit very stochastic behavior *on some subset* of the surface—possibly a set of Lebesgue measure zero. Thus, the stochastic behavior of a surface diffeomorphism with positive topological entropy may not be *physically observable*.

In this note, we observe that there exists an open and dense subset of C^{∞} symplectic (area-preserving) diffeomorphisms (symplectomorphisms) on S^2 having positive topological entropy, and we observe a related result

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for symplectomorphisms of the 2-torus. The symplectic hypothesis is essential, for the Morse–Smale diffeomorphisms of S^2 form an open subset of diffeomorphisms with zero topological entropy.[9] The proofs are easy applications of Pixton's Theorem[8] and Oliveira's Theorem.[7]

Let M^2 denote a C^∞ compact surface. A symplectic form (or area form) ω on M^2 is a smooth positive differential two-form. Denote by $Diff_\omega^\infty(M^2)$ the set of symplectomorphisms of M^2 equipped with the Whitney topology, i.e., the diffeomorphisms f of M^2 which preserve the symplectic form ω, i.e., $f^*\omega = \omega$.

We now state our main result:

Theorem 1. There exists an open and dense subset of maps $f \in Diff_\omega^\infty(S^2)$ which have positive topological entropy.

Theorem 1 is also true for $Diff_\omega^2(S^2)$ or even $Diff_\omega^{1+\alpha}(S^2)$. Applying Katok's result[2] that a surface diffeomorphism with positive topological entropy contains horseshoes (which carry *most* of the entropy) to Theorem 1 yields the following corollary:

Corollary 1. There exists an open and dense subset of maps $f \in Diff_\omega^\infty(S^2)$ which have infinitely many hyperbolic periodic points and an exponential growth rate of (hyperbolic) periodic points.

Proof of Theorem 1. The heart of the proof is the following theorem of Pixton:

Theorem (Pixton[8]). A residual subset of $Diff_\omega^\infty(S^2)$ has the property that the stable and unstable manifolds of every hyperbolic periodic point intersect transversely.

Note that Pixton's Theorem makes no claim about the existence of hyperbolic periodic points. The openness statement in Theorem 1 immediately follows from the C^1 structural stability of hyperbolic sets.[10]

To prove the density statement, let $f: S^2 \to S^2$ be a symplectomorphism having the generic property in Pixton's Theorem, i.e., that every hyperbolic periodic point has transverse homoclinic points. Choose a neighborhood $U \subset Diff_\omega^\infty(S^2)$ of f. While the map f need not have a fixed point, it follows from the Lefschetz fixed-point theorem that f has a periodic point x.[1] If x is hyperbolic, by hypothesis the stable and unstable manifolds of x intersect transversely and thus f has positive topological entropy.[11] Suppose x is elliptic (or parabolic). By studying the Birkhoff normal symplectic perturbations, Moser[6] showed that one can find a symplectomorphism $g \in U$ having a *hyperbolic* periodic point y near x. Moser's result was actually stated for real analytic maps, but several authors (see, for instance, ref. 8) have observed that a simple modification of his argument yields C^r versions of the result for all r. Since hyperbolic period points are C^1 structurally stable, Pixton's Theorem yields a symplectomorphism $h \in U$ with a hyperbolic fixed point z near y such that the stable and unstable manifolds of z intersect transversely. Hence h has positive topological entropy. ∎

Oliveira[7] generalized Pixton's Theorem to symplectomorphisms of the 2-torus $\mathbb{T}^2$. The extension to surfaces of higher genus is unknown. The same argument as in Theorem 1 would prove the existence of an open and dense subset of symplectomorphisms of $\mathbb{T}^2$ which have positive topological entropy, *provided* one knew that a dense set of symplectomorphisms of $\mathbb{T}^2$ has a periodic point. This is a weak form of the C^∞ closing lemma on $\mathbb{T}^2$ and has not yet been proved. The irrational translation on $\mathbb{T}^2$ is an example of a symplectomorphism of $\mathbb{T}^2$ without periodic points. However, it follows from the Lefschetz fixed-point theorem[12] that any diffeomorphism $f: \mathbb{T}^2 \to \mathbb{T}^2$ for which $f^*: H^1(\mathbb{T}^2) \to H^1(\mathbb{T}^2)$ does not have $+1$ as an eigenvalue has a fixed point, where $H^1(\mathbb{T}^2)$ denotes the first cohomology group of $\mathbb{T}^2$. The set of symplectomorphisms with this property clearly forms a *large* open set of symplectomorphisms.

Theorem 2. In the open set of diffeomorphisms $f: \mathbb{T}^2 \to \mathbb{T}^2$ for which $f^*: H^1(\mathbb{T}^2) \to H^1(\mathbb{T}^2)$ does not have $+1$ as an eigenvalue, there exists an open and dense subset of symplectic diffeomorphisms which have positive topological entropy.

It follows that in every connected component of $Diff_\omega^\infty$ with one exception, there exists an open and dense subset of symplectic diffeomorphisms which have positive topological entropy.

It should be pointed out that in *many* cases, there already exists a result stronger than Theorem 2. Let A be an element of $SL(2, \mathbb{Z})$ and f a diffeomorphism of $\mathbb{T}^2$ inducing A in homology. Then one has a short list of possibilities:

1. Both eigenvalues of A are real and of modulus different from 1. In this case it easily follows from Manning's *entropy inequality*[5] between entropy and the log of the spectral radius of A that f *always* has positive topological entropy.

2. Eigenvalues of A are roots of unity of order 1, 2, 3, 4, or 6. If, as in Theorem 2, we exclude 1 as an eigenvalue, we are left with only finitely many conjugacy classes in $SL(2, \mathbb{Z})$ for which Theorem 2 is nontrivial. These classes are just conjugacy classes of elements of finite order in $SL(2, \mathbb{Z})$. In other words, Theorem 2 yields new information for finitely many connected components of $Diff_\omega(\mathbb{T}^2)$.

We leave the reader with two intriguing open questions:

1. Is it true that an open and dense set of symplectomorphisms on *every* surface, or more generally, every smooth compact symplectic manifold has positive topological entropy?

2. A more refined invariant measuring the complexity of the orbit structure for a symplectic map is the measure-theoretic entropy (with respect to the Lebesgue measure induced by the symplectic form). Metric entropy gives the exponential growth rate of the statistically significant orbits. If a map has positive metric entropy, then it exhibits very stochastic behavior on a set of positive Lebesgue measure. Recall that a map with positive topological entropy may exhibit stochastic behavior on a set of Lebesgue measure zero. Is it true that an open and dense set of symplectic diffeomorphisms on every surface, or more generally, every smooth compact symplectic manifold has positive metric entropy? The KAM theorem implies that the set of ergodic symplectomorphisms is not dense.

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